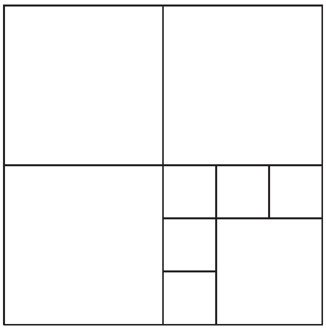
INTERNATIONAL MATHEMATICS

TOURNAMENT OF TOWNS

JUNIOR PAPER: YEARS 8, 9, 10

Tournament 31, Northern Autumn 2009 (O Level): Solutions

1. The diagram below shows that a 6×6 square can be cut into one 2×2 square, three 3×3 squares and five 1×1 squares.



(A Liu)

2. Suppose to the contrary that no two weights in the same pan differ in mass by exactly 20 grams. Then in the right pan, we must have put in exactly one weight from each of the following ten pairs: (1,21), (3,23), . . . , (19,39).

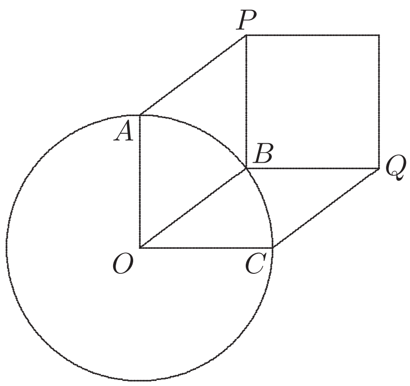
The total mass in the right pan is $1+3+\cdots+19+20k = 100+20k$, where k is the number of times we chose the heavier weight from a pair. This is a multiple of 4. Similarly, the total mass in the left pan is $2 + 4 + \cdots + 20 + 20h = 110 + 20h$, where h is the number of times we chose the heavier weight from a pair. This is not a multiple of 4.

We have a contradiction as the two pans cannot possibly balance.

(A Liu)

3. Alternative 1

Let O be the centre of the circle, A, B and C be the points of contact with the circle of three squares in order, and P and Q be the common vertices of these squares. Call OA, OB and OC the root canals of the respective squares.



Then $OAPB$ and $OBQC$ are rhombi. Moreover, $\angle PBQ = 90°$. Hence $\angle AOC = 90°$. This means that every two alternate root canals are perpendicular.

It follows that there must be 8 root canals, and the last square must have a common vertex with the first.

(A Liu)

Alternative 2

Since B, which lies on the circle, is a vertex of a square, points P, Q and O lie on the circle with radius 5 cm and centre at the point B. Hence, $\angle POQ = \frac{1}{2}\angle PBQ = 45°$. This means that every attached square can be seen at the angle 45° from the centre O of a cardboard disc and borders of the neighbouring angles coincide. Therefore, Petya can attach $\dfrac{360°}{45°} = 8$ squares and the last square must have a common vertex with the first.

(A.Shapovalov and L.Mednikov)

4. In six attempts, try entering 0123456, 0234561, 0345612, 0456123, 0561234 and 0612345. Since the correct password uses 7 different digits, it must use at least 3 of the digits 1, 2, 3, 4, 5 and 6. At most one of these 3 can be in the first place. The other 2 must match one of our attempts.

(A Liu)

5. Pretend that the 2000 people are seated at a round table, evenly spaced. Each invites the next 1000 people in clockwise order. Then only two people who are diametrically opposite to each other become friends. This shows that the number of pairs of friends may be as low as 1000.

Construct a directed graph with 2000 vertices representing the people. Each vertex is incident to 1000 outgoing arcs representating the invitations. The total number of arcs is 2000×1000. The total number of pairs of vertices is $2000 \times 1999 \div 2 = 1999 \times 1000$. Even if every pair of vertices is connected by an arc, we still have $2000 \times 1000 - 1999 = 1000$ extra arcs. These can only appear as arcs going in the opposite direction to existing arcs.

It follows that there must be at least 1000 reciprocal invitations, and therefore at least 1000 pairs of friends.

(A Liu)

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TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11, 12

Tournament 31, Northern Autumn 2009 (O Level): Solutions

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In six attempts, try entering 0123456, 0234561, 0345612, 0456123, 0561234 and 0612345. Since the correct password uses 7 different digits, it must use at least 3 of the digits 1, 2, 3, 4, 5 and 6. At most one of these 3 can be in the first place. The other 2 must match one of our attempts.

(A Liu)

2. **Alternative 1**

Suppose to the contrary the six points defining the broken line do not all lie in the same plane. Now B, C and D determine a plane, which we may assume to be horizontal. Suppose that E does not lie in this plane. Since AB is parallel to DE, A does not lie in this plane either. Since $AB \neq DE$, A and E do not lie in the same horizontal plane. Since BC is parallel to EF, F lies on the same horizontal plane as E. Since CD is parallel to FA, A lies on the same horizontal plane as F. This is a contradiction.

It follows that E also lies on the horizontal plane determined by B, C and D. Since BC is parallel to EF, F also lies in this plane, and since FA is parallel to CD, A does also.

(A Liu)

Alternative 2

Vector $\vec{AD}$ can be represented as linear combination of vectors $\vec{AB}$, $\vec{BC}$ and $\vec{CD}$ in two different ways.

$$\begin{aligned} \vec{AD} &= \vec{AB} + \vec{BC} + \vec{CD} \\ \vec{AD} &= \vec{AB} + \vec{BC} + \vec{CD} = \vec{ED} + \vec{FE} + \vec{AF} \\ &= a\vec{AB} + b\vec{BC} + c\vec{CD}, \text{where} \quad a \neq 1. \end{aligned}$$

Hence, vectors $\vec{AB}$, $\vec{BC}$ and $\vec{CD}$ lie in the same plane. So are points A, B, C, D, E and F.

(A.Shapovalov and L.Mednikov)

3. For $a = 10^{66}$, $b = 2a$, $c = 3a$ and $d = 4a$,

$$a^3 + b^3 + c^3 + d^3 = (1^3 + 2^3 + 3^3 + 4^3)(100^{33})^3 = 100^{100}.$$

(A Liu)

4. Let 1 be the side length of the regular 2009-gon $A_1A_2\ldots A_{2009}$. For indexing purposes, we treat 2010 as 1. For $1 \le k \le 2009$, let B_k be the chosen point on A_kA_{k+1} with $A_{2010} = A_1$, C_k be the image of reflection of B_k, and $d_k = A_kB_k$.

Let
$$S = d_1 + d_2 + \cdots + d_{2009}$$
and
$$T = d_1d_2 + d_2d_3 + \cdots + d_{2009}d_1.$$

Now $B_1B_2\ldots B_{2009}$ may be obtained from the regular 2009-gon by removing 2009 triangles, each with an angle equal to the interior θ angle of the regular 2009-gon, flanked by two sides of lengths $1 - d_k$ and d_{k+1}.

Hence its area is equal to that of the regular 2009-gon minus $\frac{1}{2}\sin\theta$ times
$$(1 - d_1)d_2 + (1 - d_2)d_3 + \cdots + (1 - d_{2009})d_1 = S - T.$$

Similarly, the area of $C_1C_2\ldots C_{2009}$ is equal to that of the regular 2009-gon minus $\frac{1}{2}\sin\theta$ times
$$d_1(1 - d_2) + d_2(1 - d_3) + \cdots + d_{2009}(1 - d_1) = S - T.$$

Hence these two 2009-gons have the same area.

(Olga Ivrii)

5. List all possible routes from the south capital to the north capital and index them 1, 2, 3, ... Label the first toll road on each route 1. Now the first toll road in route k may also be a later toll road in another route. Label this toll road in the other route 1, modified to $1(k)$ to keep track of why it is so labelled. All toll roads on a route between two labelled 1 are also labelled 1. This may trigger further labelling and prolong the round, but at some point, this must terminate.

Now label the first unlabelled toll road on each route 2, and so on, until all toll roads have been labelled. We continue the modification process to keep track of on which route a certain label first appears. Note that along each route, the labels on the toll roads either remain the same or increase by 1. Assign all toll roads labelled ℓ to the ℓ-th company.

We claim that each route has at least one toll road labelled 10. Assume that the highest label of a toll road on a certain route k_1 is less than 10. If each label appears exactly once on this route, then it has less than 10 toll roads, which is a contradiction. Hence some label appears more than once. Let the highest label which appears more than once be h_1, and consider the last time it appears. It must have been modified to $h_1(k_2)$ for some route k_2. We now follow k_2 until this toll road, and then switch to k_1. This combination must be one of the listed routes, say k_3.

Now the highest label of a toll road on this route is also less than 10. Hence some label appears more than once, and such a label must be less than h_1. Let the highest label which appears more than once be h_2, and consider the last time it appears. It must have been modified to $h_2(k_4)$ for some route k_4. We now follow k_4 until this toll road, and then switch to k_3. Continuing this way, we will find a route in which every label appears exactly once, and the highest label is less than 10. This is a contradiction.

(Olga Ivrii)