Solutions for the problems of the Autumn round of the 30th Tournament of Towns.

O-level, Juniors.

1. Arrange the boxes in a line so that the number of pencils in them increases from left to right. Then the first box from the left contains at least one pencil, the next one contains at least two pencils, ..., the tenth box from the left contains at least 10 pencils. Take any pencil from the first box. Since the second box contains pencils of at least two different colors, some of these pencils has color distinct from that of the chosen pencil. Take it. The third box contains pencils of at least three colors. Hence some of these pencils has color distinct from the colors of both chosen pencils. Take it. Proceeding in the same manner, we choose the required 10 pencils of different colors.

2. Subtract 50 from each given number exceeding 50. By the conditions of the problem, each of the resulting differences is distinct from any of 25 given numbers not exceeding 50. So these numbers and the differences form a set of 50 distinct positive integers not exceeding 50. Thus it contains all positive integers from 1 to 50. Their sum equals $51 \cdot 25$, hence the sum of the given numbers equals $51 \cdot 25 + 50 \cdot 25 = 101 \cdot 25 = 2525$.

3. Let B_1, B_2, B_3 be the midpoints of arcs A_1A_2, A_2A_3, A_3A_1, respectively. The area of hexagon $A_1B_1A_2B_2A_3B_3$ is the sum of the areas of quadrilaterals $OA_1B_1A_2$, $OA_2B_2A_3$, and $OA_3B_3A_1$. But each of these quadrilaterals has perpendicular diagonals, hence the area of each quadrilateral is the half-product of its diagonals. Therefore, the required sum is equal to $\frac{1}{2}OB_1 \cdot A_1A_2 + \frac{1}{2}OB_2 \cdot A_2A_3 + \frac{1}{2}OB_3 \cdot A_3A_1$. Since $OB_1 = OB_2 = OB_3 = 2$ by the conditions of the problem, this sum is numerically equal to $A_1A_2 + A_2A_3 + A_3A_1$, as required.

4. Answer. Yes, it can.

Solution. First take any three distinct positive integers such that one of them is equal to the half-sum of the remaining two; for instance, 1, 2, and 3. Their product equals 6 and so is not 2008th power of a positive integer. Multiply each of these numbers by 6^n to obtain 6^n, $2 \cdot 6^n$, $3 \cdot 6^n$. As before, one of the numbers is the half-sum of two others, and now their product equals 6^{3n+1}. It remains to choose n so that $3n + 1$ equals 2008 (or is divisible by 2008). Since 2007 is divisible by 3, we can take $3n + 1 = 2008$, that is, $n = 669$.

5. Represent the running track as the left half of a circle. We may assume that a runner at the end of the running track does not turn back but continues to run along the right half of the same circle. Thus all runners are always running along this circle. The condition that they are at the same point of the initial running track means that they are on a line perpendicular to the diameter separating the left and right halves of the circle. Suppose all runners meet (are on the corresponding line) in time t after start. Then all runners in the left and in the right halves are at the same distance x from the starting point. Each runner in the left half has covered some integer number of circles plus distance x, and each runner in the right half has to run distance x to cover some integer number of circles. Where will the runners be in time $2t$ after start? Each runner in the left half will cover some integer number of circles plus distance $2x$, and each runner in the right half will have to run distance $2x$ to cover some integer number of circles. But this means that they again will be on a line perpendicular to the diameter separating the left and right halves of the circle, because they are at the same distance (along the circle) from the starting point. Hence, on the initial running track, the runners will meet again in time $2t$, similarly in time $3t$, and so on.

O-level, Seniors.

1. Arrange the boxes in a line so that the number of cookies in them decreases from left to right. On a sheet of squared paper, draw a "staircase" in which the height of the first column (square side in width) equals the number of cookies in the first box from the left, the height of the second column equals the number of cookies in the second box, and so on. Then the staircase divides into "footsteps": the first footstep (from the left) consists of the highest columns, the second footstep consists of the columns next to the highest, and so on. The last footstep (to the right) consists of the lowest columns. The number of different integers in Alex's records is equal to the number of footsteps of this staircase (the boxes with the maximal number of cookies correspond to the highest footstep, and so on). But this number is equal to the number of different integers in Serge's records. Indeed, choosing a cooky in each box may be described as cutting off the bottom row of cells in our staircase. Therefore, when we fill up the plates with the maximal number of cookies, several rows will be removed so that the lowest footstep will disappear, and thus the number of footsteps will decrease by 1. By filling up the plates with the next to maximal number of cookies, we remove the next footstep, and so on. Hence the number of footsteps equals the number of different integers in Serge's records as required.

2. Answer: $x_1 = 1, x_2 = \dots = x_n = 0$.

Solution. Let us square the equality $\sqrt{x_1} + \sqrt{x_2 + \dots + x_n} = \sqrt{x_2} + \sqrt{x_3 + \dots + x_n + x_1}$, subtract the sum $x_1 + \dots + x_n$ from both sides, and square again. We obtain $x_1(x_2 + \dots + x_n) = x_2(x_3 + \dots + x_n + x_1)$, hence $(x_1 - x_2)(x_3 + \dots + x_n) = 0$. Since $x_1 - x_2 = 1$, we have $x_3 + \dots + x_n = 0$. Since our equations contain square roots of $x_3, \dots, x_n$, these numbers are nonnegative, and since their sum is 0, each of them is 0.

Suppose $x_2 \neq 0$, that is, $x_2 - x_3 \neq 0$. Considering the sums which contain $\sqrt{x_2}$ and $\sqrt{x_3}$ and arguing as above, we get $x_1 = 0$. Then $x_2 = -1$, but since there exists $\sqrt{x_2}$, we obtain a contradiction. Thus $x_2 = 0$, hence $x_1 = 1$, and then all conditions are satisfied.

3. Let $B_1, B_2, \dots, B_{30}$ be the midpoints of arcs $A_1A_2, A_2A_3, \dots A_{30}A_1$ respectively. The area of 60-gon $A_1B_1A_2B_2 \dots A_{30}B_{30}$ is the sum of the areas of quadrilaterals $OA_1B_1A_2, OA_2B_2A_3, \dots, OA_{30}B_{30}A_1$. But each of these quadrilaterals has perpendicular diagonals, hence the area of each quadrilateral is the half-product of its diagonals. Observe that one of these quadrilaterals can be non-convex (if the center of the circle lies outside the given 30-gon) but its area is calculated in the same way (verify this!). The required sum is then equal to $\frac{1}{2}OB_1 \cdot A_1A_2 + \frac{1}{2}OB_2 \cdot A_2A_3 + \dots + \frac{1}{2}OB_{30} \cdot A_{30}A_1$. Since $OB_1 = OB_2 = \dots = OB_{30} = 2$ by the conditions of the problem, this sum is numerically equal to $A_1A_2 + A_2A_3 + \dots + A_{30}A_1$, as required.

4. Answer. Yes, it can.

Solution. First take any arithmetic progression of five distinct positive integers, for instance, 1, 2, 3, 4, 5. Their product equals 120 and so is not 2008th power of a positive integer. Multiply each of these numbers by 120^n to obtain 120^n, $2 \cdot 120^n$, $3 \cdot 120^n$, $4 \cdot 120^n$ and $5 \cdot 120^n$. As before, the numbers form an arithmetic progression, and now their product equals 120^{5n+1}. It remains to choose n so that $5n + 1$ is divisible by 2008. This is possible, since 5 and 2008 are coprime. We need a y such that $5n + 1 = 2008y$. For instance, $y = 2$ and $n = 803$ fit. Then the product is 2008th power of 120^2.

5. We may assume that our rectangles are drawn on an infinite sheet of squared paper. Divide it into squares 2×2 and mark the cells in each square by 1, 2, 3, 4 clockwise starting from the upper left corner. Since both sides of each rectangle are of odd length, its corner cells are marked by the same number. Let us number four different colors by 1, 2, 3, 4 and paint each rectangle with the color whose number marks the corner cells. It is readily seen that the numbers in the corners of any two adjacent rectangles are distinct.